Acknowledgements

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Centroids of Areas (continued)

Example: Problem 7.2 in the textbook.

Determine the $x$-coordinate of the centroid.

Strategy: Use the formula based on integration.

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\[ \bar{x} = \frac{\int_A x \, dA}{\int_A dA} \]

What is \(dA\)?

\[ dA = (1 - x^2) \, dx \]

\[ \int_A x \, dA = \int_0^1 x(1 - x^2) \, dx \]
\[ = \int_0^1 (x - x^3) \, dx \]
\[ = \left[ \frac{1}{2} x^2 - \frac{1}{4} x^4 \right]_0^1 \]
\[ = \frac{1}{2} - \frac{1}{4} \]
\[ = \frac{1}{4} \]

\[ \int_A dA = \int_0^1 (1 - x^2) \, dx \]
\[ = \left[ x - \frac{1}{3} x^3 \right]_0^1 \]
\[ = 1 - \frac{1}{3} \]
\[ = \frac{2}{3} \]
\[ \bar{x} = \frac{\int x \, dA}{\int dA} \]

\[ = \frac{1}{4} \]

\[ = \frac{2}{3} \]

\[ = \frac{3}{8} \]

Go through Examples 7.1 and 7.2 in the textbook on your own.

For symmetric areas, the centroid lies on the axis of symmetry. This can simplify calculations greatly. Consider examples.
Composite Areas

Suppose you are asked to determine the centroid of the following area:

Note that it can be subdivided into regions of simpler geometry:
Correspondingly, we can split the integral:

\[
\bar{X} = \frac{\int_A x \, dA}{\int_A dA} = \frac{\int_{A_1} x \, dA + \int_{A_2} x \, dA + \int_{A_3} x \, dA}{\int_{A_1} dA + \int_{A_2} dA + \int_{A_3} dA}
\]

For the individual areas:

\[
\bar{X}_1 = \frac{\int_{A_1} x \, dA}{\int_{A_1} dA} = \frac{1}{A_1} \int_{A_1} x \, dA \quad \Leftrightarrow \quad \int_{A_1} x \, dA = \bar{X}_1 A_1
\]

\[
\int_{A_2} x \, dA = \bar{X}_2 A_2
\]

\[
\int_{A_3} x \, dA = \bar{X}_3 A_3
\]

Plug these in to Eq. (#):

\[
\bar{X} = \frac{\int_{A_1} x \, dA + \int_{A_2} x \, dA + \int_{A_3} x \, dA}{A_1 + A_2 + A_3} = \frac{\bar{X}_1 A_1 + \bar{X}_2 A_2 + \bar{X}_3 A_3}{A_1 + A_2 + A_3}
\]
More generally, the $x$-coordinate of the centroid of a composite area is

$$
\bar{x} = \frac{\sum \bar{x}_i A_i}{\sum A_i}
$$

By the same argument, the $y$-coordinate is

$$
\bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i}
$$

We can easily work out the centroid of a composite area from the centroids of its parts:

Example 7.3 goes through this problem in detail. Work through it on your own.
Example: Problem 7.32 from the textbook.

Determine the coordinates of the centroid.

Strategy: Subdivide into 4 regions (3 rectangles + 1 quarter circle) and use the formula for composite areas.

Centroids of simple areas are tabulated in Appendix B of the textbook.
\bar{x}_1 = 10 \text{ in}, \quad \bar{y}_1 = 20 \text{ in}, \quad A_1 = 800 \text{ in}^2

\bar{x}_2 = 10 \text{ in}, \quad \bar{y}_2 = 40 \text{ in} + \frac{30 \text{ in}}{2} = 55 \text{ in}, \quad A_2 = (30 \text{ in})(20 \text{ in}) = 600 \text{ in}^2
\[ \bar{x}_3 = 20 \text{ in} + \frac{30 \text{ in}}{2} = 35 \text{ in}, \quad \bar{y}_3 = 20 \text{ in}, \quad A_3 = (30 \text{ in})(40 \text{ in}) = 1200 \text{ in}^2 \]

\[ \bar{x}_4 = 20 \text{ in} + \frac{4R}{3\pi} = 20 \text{ in} + \frac{4(30 \text{ in})}{3\pi} = 32.732 \text{ in} \]

\[ \bar{y}_4 = 40 \text{ in} + \frac{4R}{3\pi} = 40 \text{ in} + \frac{4(30 \text{ in})}{3\pi} = 52.732 \text{ in} \]
\[ A_4 = \frac{\pi R^2}{4} = \frac{\pi (30 \text{ in})^2}{4} = 706.858 \text{ in}^2 \]

\[ \bar{x} = \frac{\sum \bar{x}_i A_i}{\sum A_i} \]

\[ = \frac{(10 \text{ in})(800 \text{ in}^2) + (10 \text{ in})(600 \text{ in}^2) + (35 \text{ in})(1200 \text{ in}^2) + (32.73 \text{ in})(706.86 \text{ in}^2)}{800 \text{ in}^2 + 600 \text{ in}^2 + 1200 \text{ in}^2 + 706.86 \text{ in}^2} \]

\[ = 23.9 \text{ in} \]

\[ \bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i} \]

\[ = \frac{(20 \text{ in})(800 \text{ in}^2) + (55 \text{ in})(600 \text{ in}^2) + (20 \text{ in})(1200 \text{ in}^2) + (52.73 \text{ in})(706.86 \text{ in}^2)}{800 \text{ in}^2 + 600 \text{ in}^2 + 1200 \text{ in}^2 + 706.86 \text{ in}^2} \]

\[ = 33.3 \text{ in} \]

\[ \bar{x} = 23.9 \text{ in}, \quad \bar{y} = 33.3 \text{ in} \]