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H_-/H_∞ structural damage detection filter design using an iterative linear matrix inequality approach

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Abstract
The existence of damage in different members of a structure can be posed as a fault detection problem. It is also necessary to isolate structural members in which damage exists, which can be posed as a fault isolation problem. It is also important to detect the time instants of occurrence of the faults/damage. The structural damage detection filter developed in this paper is a model-based fault detection and isolation (FDI) observer suitable for detecting and isolating structural damage. In systems, possible faults, disturbances and noise are coupled together. When system disturbances and sensor noise cannot be decoupled from faults/damage, the detection filter needs to be designed to be robust to disturbances as well as sensitive to faults/damage. In this paper, a new H_-/H_∞ and iterative linear matrix inequality (LMI) technique is developed and a new stabilizing FDI filter is proposed, which bounds the H_∞ norm of the transfer function from disturbances to the output residual and simultaneously does not degrade the component of the output residual due to damage. The reduced-order error dynamic system is adopted to form bilinear matrix inequalities (BMIs), then an iterative LMI algorithm is developed to solve the BMIs. The numerical example and experimental verification demonstrate that the proposed algorithm can successfully detect and isolate structural damage in the presence of measurement noise.

1. Introduction

The existence of structural damage in engineering structures, such as tall buildings, long-span bridges, offshore platforms, etc, will greatly influence the overall performance of the system or may even lead to disastrous consequences. Therefore, detecting structural damage caused by earthquake, impact, or explosion immediately after the event or monitoring long-term deterioration due to natural and non-natural hazards is necessary (Dharap *et al* 2006, Koh *et al* 2005a, 2005b).

There are many fault detection and isolation (FDI) methods. Among them, the Beard–Jones detection (BJDT) filter has gained much attention in the past few decades. In their pioneering work done in the early seventies, Beard (1971) and Jones (1973) found that with the proper choice of detection filter gains, the output residual or error function will have directional characteristics that can be easily associated with different faults. Douglas (1993), Douglas and Speyer (1996), (1999) developed the BJDT filter in a geometric framework and presented a method for reducing the effect of system disturbances and sensor noise. This geometric interpretation of the BJDT filter is adopted in this paper.

The mathematical model used in the detection filter design is never known exactly. The presence of system disturbances and sensor noise make robustness the most fundamental problem in the model-based FDI design. Recently, the H_∞ technique, which is well established in control systems, has been used to design robust FDI filters (Chen and Patton 1999, Nobrega *et al* 2000). It is often the nature of systems that the possible faults and disturbances are coupled together, so that

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the H_∞ bounded detection filter will reduce the component of the output residual due to both system disturbances and faults/damage. Such a reduction of the output residual does not adequately detect the occurrence of faults/damage, which is unacceptable. The performance of an observer-based FDI filter should, therefore, be measured via a suitable trade-off between robustness and sensitivity, i.e., the objective of fault detection filter design is to reduce the component of the output residual due to system disturbances and sensor noise without simultaneously degrading/reducing the component of the output residual due to damage. This new objective leads to the development of the so-called H_-/H_∞ method (Rank and Niemann 1999). The concept of the H_- measure was proposed by Chen and Patton (1999); it is defined as the smallest nonzero singular value of the transfer function matrix from faults to the output residual at the particular frequency, $\omega = 0$. Although this H_- measure is not a true worst-case measure, it corresponds to the case where the fault detection observer reaches its steady state, a situation where an observer becomes really useful for providing estimation/detection information (Chen and Patton 1999). There is a need to develop an H_-/H_∞ fault detection filter which can isolate different faults/damage in structural systems in the presence of disturbances and sensor noise. This would facilitate structural/vehicle health monitoring and corrective action in the form of repair or adaptive control in the case of smart structures. One potential application that the authors have been investigating is the application of such methods in fault/damage detection of the international space station, which is the subject of a separate study.

In this paper, the H_-/H_∞ FDI filter is designed by developing a new iterative LMI (ILMI) algorithm. The detection filter for the reduced-order system (Douglas 1993, Douglas and Speyer 1996, 1999) is considered. The bounded H_∞ norm of the transfer function from disturbances to the output residual is represented by linear matrix inequalities based on the bounded real lemma. Meanwhile, the H_- measure of the transfer function from the fault to the output residual is derived and represented in the form of linear matrix inequalities. The combined linear matrix inequalities (LMIs) have a bilinear form so the ILMI algorithm has to be used in order to solve them. The numerical and experimental examples presented demonstrate the capability of the presented algorithm, i.e., the detection filter obtained is not only robust to disturbances, but also sensitive to faults/damage.

2. Preliminary definition

Throughout this paper, $A > 0$ (or $A < 0$) denotes a symmetric positive (or negative) definite matrix. All matrices, if their dimensions are not explicitly stated, are assumed to be compatible. I denotes the identity matrix with appropriate dimensions. The L_2 norm of a vector function $h(t) \in L_2(0,\infty)$ is defined as

$$\|h(t)\|_2 = \sqrt{\int_0^\infty h^{\mathrm{T}}(t)h(t)\,\mathrm{d}t}. \tag{1}$$

The H_∞ norm and H_- measure of a transfer function $G(s) \in RH_\infty$ are denoted by

$$\|G(s)\|_\infty = \sup_{\omega\in R}\max_{x\neq 0}\frac{\|G(\mathrm{j}\omega)x\|_2}{\|x\|_2} = \sup_{\omega\in R}\bar{\sigma}(G(\mathrm{j}\omega)) \tag{2}$$

$$\|G(s)\|_- = \min_{x\neq 0}\frac{\|G(\mathrm{j}0)x\|_2}{\|x\|_2} = \underline{\sigma}(G(\mathrm{j}0)) \tag{3}$$

where RH_∞ is the real-rational subset of H_∞; $\bar{\sigma}$ and $\underline{\sigma}$ represent the largest and smallest singular values of the matrix $G(s)$, respectively. Notice that the H_- measure is evaluated at $\omega = 0$, which means that the fault detection observer reaches its steady state.

Typically, the H_∞ norm of the system can be computed by an iterative procedure (Doyle *et al* 1989). Recently, the LMIs technique has been widely used in computing the H_∞ norm of a system. Given a linear time-invariant system as follows:

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du,\end{aligned} \tag{4}$$

then the H_∞ norm of the transfer function $G_{yu}(s)$ from input to output is less than γ if and only if there exists a matrix $P > 0$ satisfying (Bounded Real Lemma) (Boyd *et al* 1994, Skelton *et al* 1998)

$$\begin{bmatrix} PA + A^{\mathrm{T}}P & PB & C^{\mathrm{T}}\\ B^{\mathrm{T}}P & -\gamma I & D^{\mathrm{T}}\\ C & D & -\gamma I\end{bmatrix} < 0. \tag{5}$$

The definition of the H_- measure in (3) shows that for any $x \neq 0$, $\|Ax\| \geqslant \|A\|_-\|x\|$. So, the following inequality is satisfied:

$$\begin{aligned}\|AB\|_- &= \min_{\|x\|\neq 0}\frac{\|ABx\|}{\|x\|} \geqslant \min_{\|x\|\neq 0}\frac{\|A\|_-\cdot\|Bx\|}{\|x\|}\\ &\geqslant \min_{\|x\|\neq 0}\frac{\|A\|_-\cdot\|B\|_-\cdot\|x\|}{\|x\|} = \|A\|_-\cdot\|B\|_-,\end{aligned} \tag{6}$$

i.e.,

$$\|AB\|_- \geqslant \|A\|_-\cdot\|B\|_-. \tag{7}$$

3. Detection filter problem

A linear time-invariant (LTI) dynamic system with additive faults/damage can be modeled by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_u u(t) + \sum_{i=1}^{q} F_i m_i(t)\\ y(t) &= Cx(t)\end{aligned} \tag{8}$$

where $x \in \mathcal{X}$ is an $n\times 1$ state vector, $y \in \mathcal{Y}$ is an $m\times 1$ measurement vector, A is an $n\times n$ system state transmission matrix, B_u is an $n\times r$ input influence matrix, $u \in \mathcal{U}$ is an $r\times 1$ input signal, and C is an $m\times n$ output influence matrix. $F_i \in \mathcal{F}_i$ is an $n\times 1$ fault direction vector, $i = 1, 2, \ldots, q$, q is the number of fault directions and $m_i(t)$ is the ith arbitrary scalar function of time. When no faults occur, $m_i(t) = 0$. The fault direction F_i can be used to model actuator, sensor or structural member fault (Douglas 1993, Douglas and Speyer

1996, 1999). The system is assumed to be observable. A full-order observer is designed as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_u u(t) - L\left(y(t) - C\hat{x}(t)\right)$$
$$\hat{y}(t) = C\hat{x}(t) \qquad (9)$$

where $L \in \mathbf{R}^{n\times m}$ is the observer gain matrix.

The state estimation error is defined as $\varepsilon(t) = \hat{x}(t) - x(t)$, and the output residual is defined as $r(t) = \hat{y}(t) - y(t)$; then the error system dynamics can be rewritten as

$$\dot{\varepsilon}(t) = (A + LC)\,\varepsilon(t) - \sum_{i=1}^{q} F_i m_i(t)$$
$$r(t) = C\varepsilon(t). \qquad (10)$$

In order to detect and isolate different faults, Douglas (1993) defined the following detection filter problem.

Definition 1 (Detection filter problem). Given the LTI system (8), the detection problem is to find a set of subspaces $\mathcal{T}_i \subseteq \mathcal{X}$, $i = 1, 2, \ldots, q$, such that the following three conditions are satisfied:

$$(A + LC)\mathcal{T}_i \subseteq \mathcal{T}_i \qquad \text{subspace invariance} \qquad (11)$$

$$\mathcal{F}_i \subseteq \mathcal{T}_i \qquad \text{fault inclusion} \qquad (12)$$

$$C\mathcal{T}_i \cap \sum_{j\neq i}^{q} C\mathcal{T}_j = \varnothing \qquad \text{output separability} \qquad (13)$$

where $\mathcal{T}_1, \mathcal{T}_2, \ldots$, and $\mathcal{T}_q$ are usually chosen as a set of mutually detectable, minimal unobservability subspaces or detection spaces in order to ensure stability (Douglas and Speyer 1999). The minimal unobservability subspace $\mathcal{T}_i$ includes the minimal (C, A)-invariant subspace $\mathcal{W}_i$ and the subspace $\mathcal{V}_i$ spanned by the invariant zero directions of the triple (C, A, F_i), where the invariant zeros are assumed to be distinct. Suppose W_i spans the subspace $\mathcal{W}_i$ and $\dim(\mathcal{F}_i) = 1$; then W_i can be generated as

$$W_i = \{\, F_i \quad AF_i \quad \cdots \quad A^{k_i}F_i \,\} \qquad (14)$$

where k_i is the minimum integer such that $CA^{k_i}F_i \neq 0$, i.e.,

$$CF_i = CAF_i = \cdots = CA^{k_i-1} = 0, \qquad \text{but } CA^{k_i}F_i \neq 0.$$

The subspace $\mathcal{V}_i$ spanned by V_i includes invariant zeros z_{ik} and zero directions v_{ik} of the system triple (C, A, F_i), which satisfies the following equations (Douglas 1993):

$$(A + F_i K_i)v_{ik} = z_{ik}v_{ik}$$
$$Cv_{ik} = 0 \qquad (15)$$

where K_i is some matrix with compatible dimensions.

Then, the subspace $\mathcal{V}_i$ can be constructed from $V_i = [v_{i1}, v_{i2}, \ldots, v_{ip}]$, where p is the number of invariant zero directions with respect to the given fault direction F_i. Since the invariant zeros are distinct, $v_{i1}, v_{i2}, \ldots$, and v_{ip} are linearly independent. Assume that $v_{i1}, v_{i2}, \ldots, v_{ig}$ are not included in the subspace spanned by the column space of W_i, where $0 \leqslant g \leqslant p$. Thus, the detection space $\mathcal{T}_i$ becomes

$$T_i = [\, v_{ig} \quad \cdots \quad v_{i1} \quad F_i \quad AF_i \quad \cdots \quad A^{k_i}F_i \,]. \qquad (16)$$

Since $\mathcal{T}_i$ is a (C, A)-invariant subspace, $(A + LC)T_i = T_iA_{T_i}$, where matrix A_{T_i} has the compatible dimensions. Combining (14), (15) and (16) leads to

$$\begin{aligned}
(A + LC)\,T_i &= (A + LC) \\
&\quad \times [\, v_{ig} \quad \cdots \quad v_{i1} \quad F_i \quad AF_i \quad \cdots \quad A^{k_i}F_i \,] \\
&= [\, Av_{ig} \quad \cdots \quad Av_{i1} \quad AF_i \quad A^2F_i \quad \cdots \quad (A + LC)A^{k_i}F_i \,] \\
&= T_i \left[\begin{array}{ccc:cccccc}
z_{ig} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \beta_{ig} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & z_{i1} & 0 & 0 & 0 & \cdots & 0 & \beta_{i1} \\
\hdashline
y_{ig} & \cdots & y_{i1} & 0 & 0 & 0 & \cdots & 0 & \alpha_0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & \alpha_1 \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & \alpha_2 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \alpha_{\mu_i-1} \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & \alpha_{\mu_i}
\end{array}\right] \\
&= T_iA_{T_i}
\end{aligned} \qquad (17)$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{\mu_i}$ and $\beta_{i1}, \ldots, \beta_{ig}$ are unknown parameters, which are defined as

$$(A + LC)A^{k_i}F_i = T_i\,[\, \beta_{ig} \quad \cdots \quad \beta_{i1} \quad \alpha_0 \quad \alpha_1 \quad \cdots \quad \alpha_{\mu_i} \,]^{\mathrm{T}}. \qquad (18)$$

Once a set of detection spaces $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_q$ has been found, the parameterization of detection filter gain L is shown in theorem 1.

Theorem 1 (detection filter parameterization). *Let $\mathcal{T}_1$, $\mathcal{T}_2$, …, $\mathcal{T}_q$ be a set of (C, A)-invariant subspaces that solve the detection filter problem and let T_i: $\mathcal{T}_i \mapsto \mathcal{X}$ be the insertion map. Let P_i, $\hat{F}_i$, H_i and $\tilde{H}_i$ associated with $\mathcal{T}_i$ be defined as follows. Let P_i: $\mathcal{T}_i \mapsto \mathcal{T}_i$ be any projection where* $\mathrm{Ker}(P_i) = \mathrm{Ker}(C\mathcal{T}_i)$ *and let $\hat{F}_i$ decompose P_i as $\hat{F}_i\hat{F}_i^{\mathrm{T}} = P_i$ and $\hat{F}_i^{\mathrm{T}}\hat{F}_i = I$. Let H_i: $\mathcal{Y} \mapsto \mathcal{Y}$ be another projection where* $\mathrm{Im}(H_i) = C\mathcal{T}_i$ *and let $\tilde{H}_i$ be the associated natural projection that satisfies $C\mathcal{T}_i\hat{F}_i\tilde{H}_i = H_i$ and $\tilde{H}_iC\mathcal{T}_i\hat{F}_i = I$. Specify the kernel of H_i and $\tilde{H}_i$ as*

$$\sum\nolimits_{j\neq i}^{q} C\mathcal{T}_j \subseteq \mathrm{Ker}(H_i) = \mathrm{Ker}(\tilde{H}_i).$$

Also, define the projection

$$H_0 = \left(I - \sum\nolimits_{i=1}^{q} H_i\right)$$

and the associated natural projection $\tilde{H}_0$. Finally, define a set of maps

$$\underline{L}(\mathcal{T}_i) = \{L : \mathcal{Y} \mapsto \mathcal{X} \,|\, (A + LC)\mathcal{T}_i \subseteq \mathcal{T}_i\}.$$

Then $L \in \cap_{i=1}^{q}\underline{L}(\mathcal{T}_i)$ if and only if

$$L = \sum_{i=1}^{q}\left(-AT_i\hat{F}_i + T_ia_i\right)\tilde{H}_i + b\tilde{H}_0 \qquad (19)$$

for some b: $\mathrm{Im}\,H_0 \mapsto \mathcal{X}$ *and $a_i = A_{T_i}\hat{F}_i$ $(i = 1, 2, \ldots, q)$.*

Proof. The proof for theorem 1 has been presented by Douglas (1993).

As long as a set of (C, A)-invariant subspaces solve the detection problem (definition 1), the observer gain L can be parameterized by equation (19). The solution is not unique due to the different a_i and b values. They also provide us with the freedom to find a better observer gain L such that the noise effect is reduced. Notice that $CT_i = [0, 0, \ldots, 0, CA^{k_i}F]$, and $\mathrm{Ker}(P_i) = \mathrm{Ker}(C\mathcal{T}_i)$, $\hat{F}_i\hat{F}_i^{\mathrm{T}} = P_i$, $\hat{F}_i^{\mathrm{T}}\hat{F}_i = I$; it is easy to find that $\hat{F}_i = [0, 0, \ldots, 0, 1]^{\mathrm{T}}$. Thus

$$
\begin{aligned}
a_i &= A_{T_i}\hat{F}_i \\
&= \left[\begin{array}{ccc:ccccc c}
z_{ig} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \beta_{ig} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & z_{i1} & 0 & 0 & 0 & \cdots & 0 & \beta_{i1} \\
\hdashline
y_{ig} & \cdots & y_{i1} & 0 & 0 & 0 & \cdots & 0 & \alpha_0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & \alpha_1 \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & \alpha_2 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \alpha_{\mu_i-1} \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & \alpha_{\mu_i}
\end{array}\right]
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \beta_{ig} \\ \vdots \\ \beta_{i1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{\mu_i} \end{bmatrix}.
\end{aligned}
\tag{20}
$$

Substituting (19) into (10) yields

$$
\begin{aligned}
\dot{\varepsilon}(t) &= \left(\hat{A} + T_1a_1\tilde{H}_1C + \cdots + T_qa_q\tilde{H}_qC + b\tilde{H}_0C\right)\varepsilon(t) \\
&\quad - \sum_{i=1}^{q} F_im_i(t) \\
\tilde{z}_i(t) &= \tilde{H}_iC\varepsilon(t) \qquad i = 1, 2, \ldots, q
\end{aligned}
\tag{21}
$$

where $\tilde{z}_i$ is the system output (or the failure indicator); a_0, a_1, …, a_q and b are unknown parameters; and $\hat{A}$ is defined as

$$
\hat{A} = A + \left[\sum_{i=1}^{q}\left(-AT_i\hat{F}_i\right)\tilde{H}_i\right]C. \tag{22}
$$

If a large group of faults is considered, it may be wise to define a reduced-order system with only two unknown matrices a_i and b, which determine the dynamics of the single failure indicator $\tilde{z}_i$. Define the detection filter complementary space $\mathcal{T}_0 \subseteq \mathcal{X}$ such that $\mathcal{X} = \mathcal{T}_0 \oplus (\mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_q)$. The columns of T_i span $\mathcal{T}_i$, the columns of $T_{i0} = [T_i, T_0]$ span $\mathcal{T}_0 \oplus \mathcal{T}_i$ and the columns of $T = [T_1, \ldots, T_q, T_0]$ span $\mathcal{X}$. Let $(T)_{i0}^{-l}$ be the rows of T^{-1} such that $(T)_{i0}^{-l}T_{i0} = I$ and $(T)_{i0}^{-l}T_{j\neq i} = 0$. Define the canonical projection $\bar{P}_i = (T)_{i0}^{-l}$. Applying $\bar{P}_i$ to (21) and defining $\bar{\varepsilon}_i = \bar{P}_i\varepsilon_i$, we have the following reduced-order dynamic system:

$$
\begin{aligned}
\dot{\bar{\varepsilon}}_i(t) &= (T)_{i0}^{-l}\left(\hat{A} + T_ia_i\tilde{H}_1C + b\tilde{H}_0C\right)T_{i0}\bar{\varepsilon}_i(t) \\
&\quad - (T)_{i0}^{-l}F_im_i(t) \\
\tilde{z}_i &= \tilde{H}_iCT_{i0}\bar{\varepsilon}_i(t).
\end{aligned}
\tag{23}
$$

The above equation only has two unknown parameters: a_i and b. It is easier to use than (21) to find the desirable parameters for the ith fault detection and isolation.

Since system disturbances and sensor noise are unavoidable practically, the robust FDI filter design problem has attracted much attention recently. Robustness means that the filter has the characteristics of both noise rejection and fault sensitivity enhancement. In the following sections, the robust FDI filter design strategy is proposed using the ILMI approach. □

4. Disturbance robust detection filter problem

The LTI system with q failure modes is extended to include system disturbances and sensor noise:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_uu(t) + B_dd(t) + \sum_{i=1}^{q}F_im_i(t) \\
y(t) &= Cx(t) + D_dd(t)
\end{aligned}
\tag{24}
$$

where B_d is an $n \times n_d$ noise input influence matrix, D_d is an $m \times n_d$ noise direct transmission matrix. The input $d(t)$ includes dynamic disturbances and sensor noise and is square integrable over $[0, \infty)$. Similarly, the error dynamics can be described as

$$
\begin{aligned}
\dot{\varepsilon}(t) &= (A + LC)\,\varepsilon(t) - (B_d + LD_d)d(t) - \sum_{i=1}^{q}F_im_i(t) \\
z_i(t) &= \tilde{H}_iC\varepsilon(t) - \tilde{H}_iD_dd(t) \qquad i = 1, 2, \ldots, q.
\end{aligned}
\tag{25}
$$

Since disturbances only add forcing terms to the error dynamics, the detection filter parameterization of (19) needs no modification. Hence, the reduced-order system becomes (Douglas 1993)

$$
\begin{aligned}
\dot{\bar{\varepsilon}}_i(t) &= (T)_{i0}^{-l}\left(\hat{A} + T_ia_i\tilde{H}_1C + b\tilde{H}_0C\right)T_{i0}\bar{\varepsilon}_i(t) \\
&\quad - (T)_{i0}^{-l}\left(\hat{B}_d + T_ia_i\tilde{H}_iD_d + b\tilde{H}_0D_d\right)d(t) \\
&\quad - (T)_{i0}^{-l}F_im_i(t) \\
\tilde{z}_i &= \tilde{H}_iCT_{i0}\bar{\varepsilon}_i(t) - \tilde{H}_iD_dd(t)
\end{aligned}
\tag{26}
$$

where

$$
\hat{B}_d = B_d + \left[\sum_{i=1}^{q}\left(-AT_i\hat{F}_i\right)\tilde{H}_i\right]D_d.
$$

Notice that each reduced-order filter produces only one failure indicator $\tilde{z}_i$. A set of reduced-order filters can be designed independently. Each of them only monitors one particular fault. The system performance of the ith reduced-order filter is determined by two unknown matrices: a_i and b. We can adjust these two matrices to bound the H_∞ norm of the transfer function from disturbances to the failure indicator. This H_∞ norm problem is stated in theorem 2.

Theorem 2. *Given the system (26), the H_∞ norm of the transfer function $G_{z_i d}$ from disturbance d to z_i is less than γ if and only if there exist a matrix $P > 0$ and a scalar $\lambda > 0$ satisfying*

$$\begin{bmatrix} *_1 & *_2 & (\tilde{H}_i C T_{i0})^{\mathrm{T}} \\ *_3 & -\gamma I & (-\tilde{H}_i D_d)^{\mathrm{T}} \\ \tilde{H}_i C T_{i0} & -\tilde{H}_i D_d & -\gamma I \end{bmatrix} < 0. \tag{27}$$

where

$$*_1 = P\left[(T)_{i0}^{-l}(\hat{A} + T_i a_i \tilde{H}_1 C + b\tilde{H}_0 C)T_{i0}\right] + \left[(T)_{i0}^{-l}(\hat{A} + T_i a_i \tilde{H}_1 C + b\tilde{H}_0 C)T_{i0}\right]^{\mathrm{T}} P + 2\lambda P$$

$$*_2 = -P\left[(T)_{i0}^{-l}(\hat{B}_d + T_i a_i \tilde{H}_i D_d + b\tilde{H}_0 D_d)\right]$$

$$*_3 = -\left[(T)_{i0}^{-l}(\hat{B}_d + T_i a_i \tilde{H}_i D_d + b\tilde{H}_0 D_d)\right]^{\mathrm{T}} P.$$

Proof. Recall (5) and replace the system matrices A, B, C and D by those in (26); then (27) is obviously obtained. In (27), λ (>0) is used to place the eigenvalues of the closed-loop system to the left of $-\lambda$, which can improve the dynamic performance of the closed-loop system.

Notice that (27) is not a linear matrix inequality, but has a bilinear form. It cannot be solved directly using Matlab LMI toolbox. Assume $Q_i = P(T)_{i0}^{-l} T_i a_i$ and $Q_0 = P(T)_{i0}^{-l} b$, and then substitute them into (27); we have theorem 3. □

Theorem 3. *Given the system (26), the H_∞ norm of the transfer function $G_{z_i d}$ from disturbance d to z_i is less than γ if and only if there exists a matrix $P > 0$, Q_i and Q_0 such that*

$$\begin{bmatrix} *_4 & *_5 & (\tilde{H}_i C T_{i0})^{\mathrm{T}} \\ *_6 & -\gamma I & (-\tilde{H}_i D_d)^{\mathrm{T}} \\ (\tilde{H}_i C T_{i0}) & (-\tilde{H}_i D_d) & -\gamma I \end{bmatrix} < 0 \tag{28}$$

where

$$*_4 = P\left[(T)_{i0}^{-l}\hat{A}T_{i0}\right] + \left[(T)_{i0}^{-l}\hat{A}T_{i0}\right]^{\mathrm{T}} P + \left(Q_i \tilde{H}_i C T_{i0}\right) + \left(Q_i \tilde{H}_i C T_{i0}\right)^{\mathrm{T}} + Q_0 \tilde{H}_0 C T_{i0} + \left(Q_0 \tilde{H}_0 C T_{i0}\right)^{\mathrm{T}} + 2\lambda P$$

$$*_5 = -P(T)_{i0}^{-l}\hat{B}_d - \left(Q_i \tilde{H}_i D_d\right) - \left(Q_0 \tilde{H}_0 D_d\right)$$

$$*_6 = \left(-P(T)_{i0}^{-l}\hat{B}_d - \left(Q_i \tilde{H}_i D_d\right) - \left(Q_0 \tilde{H}_0 D_d\right)\right)^{\mathrm{T}}.$$

Remark. Given a larger initial γ, the above LMI can be solved for feasible P, Q_i and Q_0. The unknown matrices a_i and b, however, cannot be solved directly through the assumption $Q_i = P(T)_{i0}^{-l} T_i a_i$ and $Q_0 = P(T)_{i0}^{-l} b$. But, they may be obtained by the iterative algorithm as shown in section 6.

5. Fault sensitivity enhancement problem

Recall that the objective of detection filter design is not only to reduce the H_∞ norm of the transfer function from disturbances $d(t)$ to the failure indicator $z_i(t)$, but also to increase the H_- measure of the transfer function from faults $m_i(t)$ to $z_i(t)$. Recall the error dynamics representation (25)

$$\dot{\varepsilon}(t) = (A + LC)\,\varepsilon(t) - (B_d + LD_d)d(t) - \sum_{i=1}^{q} F_i m_i(t)$$
$$z_i(t) = \tilde{H}_i C\varepsilon(t) - \tilde{H}_i D_d d(t) \qquad i = 1, 2, \ldots, q. \tag{29}$$

The H_- measure of the transfer function from the fault to the failure indicator is

$$\|G(\mathrm{j}0)\|_- = \left\|\tilde{H}_i C(A + LC)^{-1} F_i\right\|_-. \tag{30}$$

Since the observer gain L is selected such that $(A+LC)$ is stable, i.e., all eigenvalues of $(A+LC)$ have negative real parts, matrix $(A + LC)$ is non-singular. Since $(A + LC)T_i = T_i A_{T_i}$, A_{T_i} is non-singular. Thus, we have

$$(A + LC)^{-1} T_i = T_i A_{T_i}^{-1}. \tag{31}$$

Recall the definition of $\mathcal{T}_i$ (16)

$$T_i = [\, v_{iq} \quad \cdots \quad v_{i1} \quad F_i \quad AF_i \quad \cdots \quad A^{k_i} F_i \,]. \tag{32}$$

So

$$F_i = T_i\,[0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0]^{\mathrm{T}} = T_i E_i. \tag{33}$$

Substituting (31) and (33) into (30) yields

$$\|G(\mathrm{j}0)\|_- = \left\|\tilde{H}_i C(A + LC)^{-1} F_i\right\|_- = \left\|\tilde{H}_i C(A + LC)^{-1} T_i E_i\right\|_- = \left\|\tilde{H}_i C T_i A_{T_i}^{-1} E_i\right\|_-. \tag{34}$$

On the basis of (7), we have

$$\|G(\mathrm{j}0)\|_- = \left\|\tilde{H}_i C T_i A_i^{-1} E_i\right\|_- \geqslant \left\|\tilde{H}_i C T_i\right\|_- \cdot \|A_{T_i}^{-1}\|_- \cdot \|E_i\|_-. \tag{35}$$

In equation (35), the proposed approach is conservative because the H_- measure is difficult to apply directly. Since $\|\tilde{H} C T_i\|_-$ and $\|E_i\|_-$ are all constant, we only need to consider $\|A_{T_i}^{-1}\|_-$. Therefore, our objective is to maximize $\|A_{T_i}^{-1}\|_-$. $\|A_{T_i}^{-1}\|_-$ is the minimum singular value of the matrix $A_{T_i}^{-1}$. Equivalently, it is the reciprocal of the maximum singular value of matrix A_{T_i}. In other words, we should minimize $\|A_{T_i}\|$ to enhance fault sensitivity. Mathematically, the 2-norm is equivalent to Frobenius norm. Thus, the objective is changed to minimize the Frobenius norm of A_i, which is defined as

$$\min \|A_{T_i}\|_F = \sqrt{\sum_{j=1}^{nn}\sum_{k=1}^{nn} A_{T_i,jk}^2} \tag{36}$$

where *nn* is the dimension of matrix A_{T_i}.

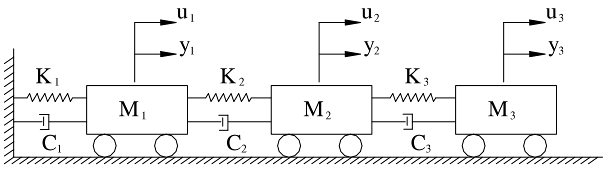

Figure 1. Three-input (u_1, u_2, u_3), three-output (y_1, y_2, y_3) spring–mass–damper system.

In (17), the only unknown entries of matrix A_{T_i} are α_0, α_1, ..., α_{μ_i}, and β_{i1}, ..., β_{ig}. Notice that $a_i = [\beta_{i1}, \ldots, \beta_{ig}, \alpha_0, \alpha_1, \ldots, \alpha_{\mu_i}]^{\mathrm{T}}$. Thus, the objective function of minimization in (36) can be simplified to

$$\min \|a_i\|_F^2 = a_i^{\mathrm{T}} a_i. \tag{37}$$

It can also be realized by the following LMI formulation:

$$\min \beta,\ \text{s.t.} \begin{bmatrix} \beta I & a_i^{\mathrm{T}} \\ a_i & \beta I \end{bmatrix} > 0. \tag{38}$$

This LMI equation can be combined with LMI (28) to satisfy our objective H_-/H_∞ requirements. As mentioned before, these LMI equations are not linear matrix inequalities; an iterative procedure is recommended.

6. The iterative LMI algorithm for the H_-/H_∞ problem

The iterative procedure developed is as follows.

Step 1: Select a large γ and substitute it into (28) to find the feasible solutions for P, Q_0 and Q_i.

Step 2: Substitute P into (27) to find the feasible solutions for a_i and γ.

Step 3: Substitute a_i into (27) to find the minimum γ_j and the corresponding matrix P_j, where j is the iteration number ($j = 1, 2, 3, \ldots$).

Step 4: Substitute P_j and γ_j into (27) and (38) to find the minimum $\beta_j = \|a_i\|$ and a_i.

Step 5: Go back to *Step 3* until $|\beta_{j+1} - \beta_j| < \varepsilon$, where ε is the error tolerance.

Step 6: Substitute a_i and γ into (28) to find P and Q_0; then at least one solution of b can be found: $b = T^+ P^{-1} Q_0$, where T^+ is the pseudo-inverse of the matrix $(T)_{i0}^{-l}$.

Remark 1. Usually, *Step 1* has feasible solutions for P, Q_i and Q_0 as long as γ is large enough. The definition of $(T)_{i0}^{-l}$ is $(T)_{i0}^{-l}[T_i\ T_0] = I$, so we have

$$(T)_{i0}^{-l} T_i = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{39}$$

Since we assume $Q_i = P(T)_{i0}^{-l} T_i a_i$, a_i may not have a solution given Q_i and P if we notice the fact of (39). Therefore, further iterative steps are necessary.

Remark 2. Substitute P which is obtained in *Step 1* into *Step 2* and find the feasible solution of a_i and Q_0. In this step, γ_j may be very large in that P is not properly selected. This problem will be considered in *Step 3*. *Step 3* uses a_i obtained in *Step 2* and solves the minimization problem for γ_j and the corresponding P_j. *Step 4* substitutes P_j and γ_j into (28) and (38) to find the minimum $\beta_j = \|a_i\|$ and a_i. *Step 5* continues *Step 3* and *Step 4* until $|\beta_{j+1} - \beta_j| < \varepsilon$.

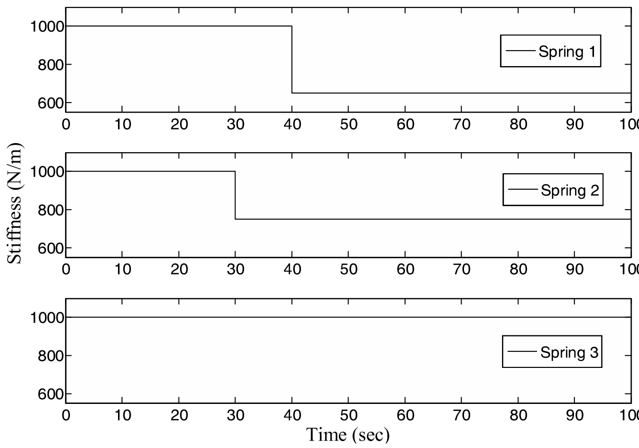

Figure 2. Stiffness variation in springs 1, 2 and 3.

Remark 3. *Step 5* iterates *Step 3* and *Step 4* to minimize γ_j and β_j. The reason is that β_j obtained in *Step 4* will not be larger than that in *Step 3*; on the other hand, γ_j obtained in *Step 3* will also not be larger than that in *Step 4*. These two values γ_j and β_j can be gradually reduced. The final values of γ and β are not the optimal values. In other words, they are not the minimal γ or β, but there is a trade-off between them. While the algorithm is not guaranteed to always find a solution, the following numerical example and experimental verification are presented to demonstrate the effectiveness of the algorithm. Actually, the practical system matrices A, B, C and D contain orderless real numbers, which make the convergence of the above iterative process easy to achieve.

7. Numerical simulation

As shown in figure 1, a three degree-of-freedom (DOF) spring–mass–damper system is used to demonstrate the robust filter design proposed in this paper. The stiffness of each spring is 1000 N m^{-1} and the mass at each node is 1 kg. Proportional Rayleigh damping is considered, i.e., $C = M + 0.001\,K$, where M, C, K are the system mass, damping and stiffness matrix, respectively. The system has three inputs and three outputs. Displacement measurement is considered in this paper. Let us suppose that spring 3 is the most reliable one and remains intact during the time history. Springs 1 and 2, however, are partially damaged during the time history. The stiffness variations in springs 1 and 2 is shown in figure 2. The stiffness of spring 1 reduces to 650 N m^{-1} in the time interval 50–100 s and the stiffness of spring 2 reduces to 750 N m^{-1} between 30 and 100 s. The stiffness matrix of the 3DOF system can be expressed as

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}. \tag{40}$$

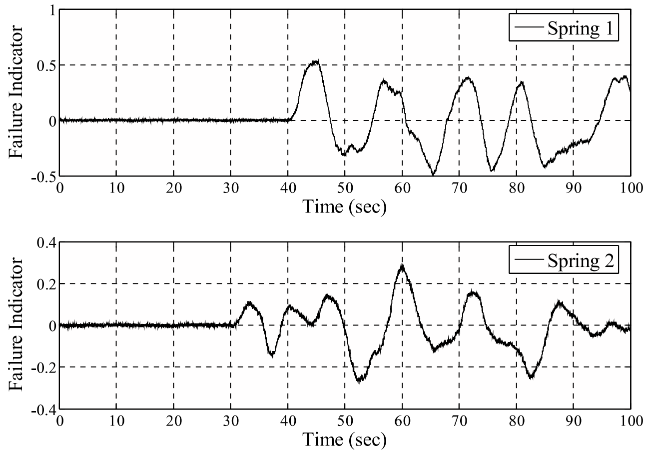

Figure 3. Isolated residual of springs 1 and 2.

Suppose the stiffness changes of the three springs are Δk_1, Δk_2 and Δk_3; then the overall stiffness matrix changes to

$$\Delta K = \begin{bmatrix} \Delta k_1 + \Delta k_2 & -\Delta k_2 & 0 \\ -\Delta k_2 & \Delta k_2 + \Delta k_3 & -\Delta k_3 \\ 0 & -\Delta k_3 & \Delta k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Delta k_1 [1 \quad 0 \quad 0] + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Delta k_2 [1 \quad -1 \quad 0] + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Delta k_3 [0 \quad 1 \quad -1]. \quad (41)$$

Thus, the state-space representation for this 3DOF system with two possible damage types in springs 1 and 2 is

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2000 & 1000 & 0 & -3 & 1 & 0 \\ 1000 & -2000 & 1000 & 1 & -3 & 1 \\ 0 & 1000 & -1000 & 0 & 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} d(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} m_1(t) \\ m_2(t) \end{Bmatrix} \quad (42)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 0.3 \\ 2 \end{bmatrix} d(t). \quad (43)$$

Clearly, $CF_1 = CF_2 = 0$, and $CAF_1 \neq 0$ and $CAF_2 \neq 0$. Because neither of the triples (C, A, F_1), (C, A, F_2) has invariant zeros, the minimal unobservability subspaces for the faults are given by the fault directions themselves, that is, $T_1 = \text{span}\{F_1, AF_1\}$, $T_2 = \text{span}\{F_2, AF_2\}$. T_0 is selected such that T_1, T_2, T_3 and T_0 span the whole space. In this paper, we select T_0 as

$$T_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 & 4 & -3 \end{bmatrix}^{\mathrm{T}}.$$

Firstly, the iterative algorithm is used to find the unknown matrices a_1 and b for fault 1. The result is $\gamma_{\min} = 2.8$, $\beta_{\min} = 0.329$, $a_1 = [-0.2532, -0.2097]^{\mathrm{T}}$ and $b = [380.1, -382.6, 0.4048, -696.0, -696.0, 762.7]^{\mathrm{T}}$. Then the iterative algorithm is used for fault 2: $\gamma_{\min} = 2.3$, $\beta_{\min} = 0.328$, $a_1 = [-0.2517, -0.2097]^{\mathrm{T}}$ and $b = [1019.0, -3.583, 1.239, 0, -1019.0, 1143.3]^{\mathrm{T}}$. Figure 3 shows residual histories where 5% rms noise with zero mean and unit variance is applied to the outputs. Comparison between the actual stiffness variation (figure 2) and the simulation results (figure 3) shows that the fault detection and isolation filter can detect and isolate different faults occurring at different times with an appropriate threshold.

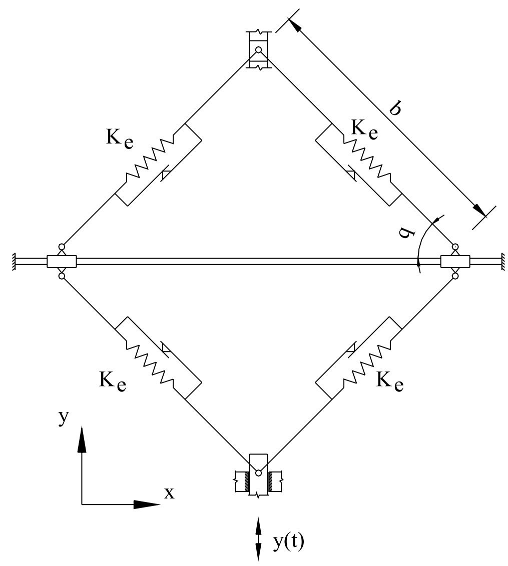

Figure 4. Analytical model of the SAIVS device.

8. Experimental verification

A semi-active independently variable stiffness (SAIVS) device developed by Nagarajaiah and Mate (1998) can vary the stiffness smoothly and continuously between maximum and minimum stiffness. Saharabudhe (2002) experimentally studied the force–displacement characteristics of a medium scaled SAIVS device. The effectiveness of the SAIVS device in reducing the seismic response of sliding base isolated buildings and bridges has been demonstrated (Saharabudhe and Nagarajaiah 2005, Nagarajaiah and Saharabudhe 2006). Nagarajaiah and Nadathur (2005) also studied the effectiveness of using a semi-active tuned mass damper (STMD) in a tall benchmark building. Nagarajaiah and Narasimhan (2007) presented a new semi-active variable damper for controlling the seismic response of smart base isolated buildings. The analytical model of the SAIVS device is shown in figure 4.

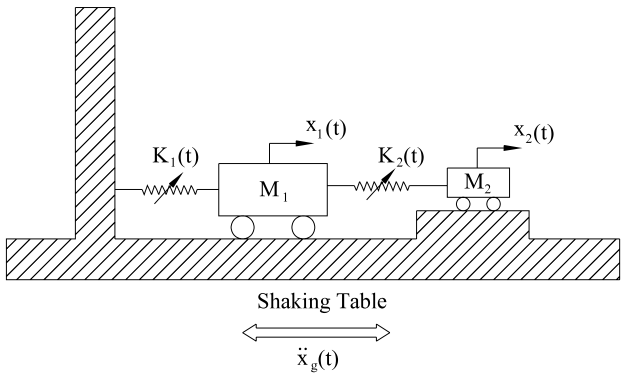

Figure 5. Schematic of a 2DOF system with SAIVS devices.

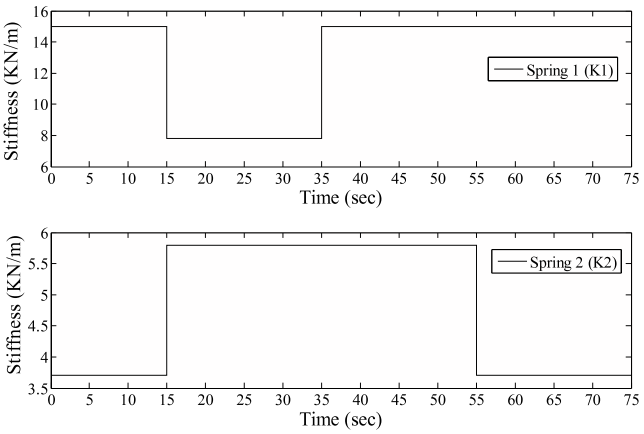

Figure 6. Stiffness changes of the SAIVS devices.

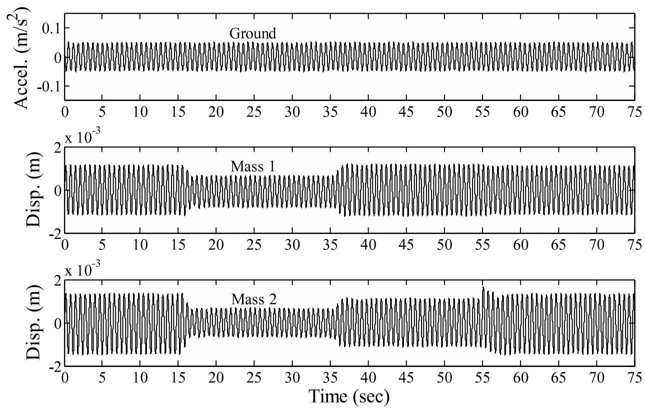

Figure 7. Time histories of the relative displacements of the first and second masses.

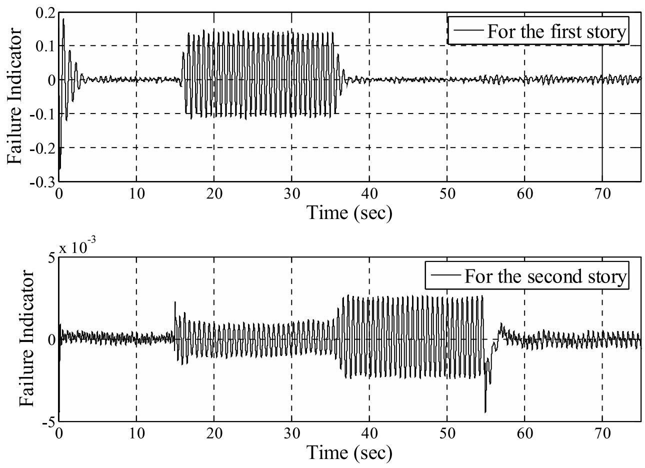

Figure 8. Fault indicators, with the H_∞/H_- method.

Springs can change the configuration by sliding in the x-direction, thus varying the effective stiffness in the y-direction. The ability of the SAIVS device to vary the stiffness of the system in real time is exploited in this paper to validate the proposed real-time structural damage detection method.

A 2DOF linear time-varying system, comprised of two SAIVS devices, is shown in figure 5. This 2DOF time-varying system was tested at Rice University, Houston, TX, USA. The stiffness of the springs was varied in real time for the 2DOF system. This system was mounted on a shake table and sinusoidal ground excitations were input into the system. The mass of the second DOF denoted by M_2 is 9.77 kg. The stiffness of the SAIVS device attached between the second DOF and first DOF denoted by K_2 was 5.8 kN m^{-1} (when fully open, $\theta = 90°$). The mass of the first DOF denoted by M_1 was 242.13 kg. The stiffness of the SAIVS device attached between the first DOF and the shake table denoted by K_1 was 15 kN m^{-1} (when fully open, $\theta = 90°$). Different damage scenarios were defined by varying the stiffness of each SAIVS device as shown in figure 6. K_1 is 15 kN m^{-1} between 0 and 15 s, 7.8 kN m^{-1} between 15 and 35 s, and 15 kN m^{-1} between 35 and 75 s; K_2 is 3.7 kN m^{-1} between 0 and 15 s, 5.8 kN m^{-1} between 15 and 55 s and 3.7 kN m^{-1} between 55 and 75 s. As shown in figure 5, the 2DOF system is excited by the sinusoidal ground excitation. The time histories of the relative displacements of the first and second masses are shown in figure 7. Due to stiffness changes of SAIVS devices, the time responses of masses 1 and 2 in the interval of changes are different from the normal counterparts. But, they are so involved that we cannot tell which SAIVS device has changed its stiffness. Therefore, it is necessary to use the proposed method for structural damage detection and isolation.

Figure 8 shows residual histories, using the measured input–output data. The damage detection results are very impressive. The noise effect is reduced, and the fault information is increased. Notice that in figure 8, the failure indicator is not close to zero at the beginning. This is because of the incorrect initial state estimation. But, they converge quickly to around zero since the observer system is stable. Although the proposed H_∞/H_- algorithm and iterative procedure are complex, the damage detection results are very successful as long as a feasible solution exists.

9. Conclusions

In this paper, the robust observer-based fault/damage detection and isolation problem is solved by developing an iterative LMI technique. The reduced-order system, which only produces one particular failure indicator, is utilized to generate the robust

FDI filter. The filter gains are designed independently for different faults/damage. The H_∞ norm of the transfer function from disturbances to residual is minimized and at the same time the H_- measure of the transfer function from fault to residual is made as large as possible. The H_- measure and H_∞ norm are computed using the iterative LMI approach developed. The examples demonstrate that the algorithm presented in this paper can detect and isolate different faults/damage robustly.

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